Cartan's structure of symmetry pseudo-group and a covering for the modified Khokhlov-Zabolotskaya equation

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Abstract. We apply Cartan's method of equivalence to find a covering for the modified Khokhlov–Zabolotskaya equation.

AMS classification scheme numbers: 58H05, 58J70, 35A30

1. Introduction

In this paper we derive a covering for the modified Khokhlov-Zabolotskaya equation [20] (or the dispersionless modified Kadomtsev-Petviashvili equation, [6])

$$u_{yy} = u_{tx} + \left(\tfrac{1}{2}\,u_x^2 - u_y\right)\,u_{xx} \tag{1}$$

from Maurer-Cartan forms (MC forms) of its symmetry pseudo-group.

Coverings [17, 18, 19] (or prolongation structures [30], or zero-curvature representations [31], or integrable extensions [1]) are of great importance in geometry of differential equations. They are a starting point for inverse scattering transformations, Bäcklund transformations, recursion operators, nonlocal symmetries and nonlocal conservation laws. Different techniques are developed for constructing coverings of partial differential equations (PDEs) in two independent variables, [30, 7, 8, 13, 21, 22, 28, 14], while in the case of more than two independent variables the problem is more difficult, see, e.g., [25, 26, 32, 29, 21, 11, 12]. In the pioneering work [20], Cartan's method of equivalence was applied to the covering problem for equations in three independent variables. One of the results of [20] is a deduction of the system

$$v_t = (v^2 - u)\,v_x - u_y - v\,u_x, \tag{2}$$

$$v_y = v\,v_x - u_x, \tag{3}$$

whose integrability conditions coincide with the Khokhlov-Zabolotskaya equation [16]

$$u_{yy} = u_{tx} + u\,u_{xx} + u_x^2. \tag{4}$$

In terms of [17, 18, 19], system (2), (3) defines an infinite-dimensional covering for equation (4). Also, a Bäcklund transformation from (4) to (1) is found in [20]: from (3) it

follows that there exists a function w such that $w_x = v$ and $w_y = \frac{1}{2}v^2 - u$; then from (2) it follows that w satisfies (1).

The present paper is an attempt to clarify the method of [20]. We apply Cartan's method of equivalence, [2]–[5], [10, 15, 27, 9], to compute MC forms for the pseudo-group of contact symmetries of (1) and then find their linear combination that gives covering equations for (1).

2. Cartan's structure theory of contact symmetry pseudo-groups of DEs

In this section, we outline the algorithm of computing MC forms for pseudo-groups of contact symmetries for DEs of the second order with one dependent variable, see details in [23, 24]. All considerations are of local nature, and all mappings are real analytic. Let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a vector bundle with the local base coordinates $(x^1, ..., x^n)$ and the local fibre coordinate u; then by $J^2(\pi)$ denote the bundle of the second-order jets of sections of π, with the local coordinates (x^i, u, u_i, u_{ij}), $i, j \in \{1, ..., n\}$, $i \le j$. For every local section $(x^i, f(x))$ of π, denote by $j_2(f)$ the corresponding 2-jet $(x^i, f(x), \partial f(x)/\partial x^i, \partial^2 f(x)/\partial x^i \partial x^j)$. A differential 1-form ϑ on $J^2(\pi)$ is called a *contact form* if it is annihilated by all 2-jets of local sections: $j_2(f)^* \vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - u_i\,dx^i$, $\vartheta_i = du_i - u_{ij}\,dx^j$, $i, j \in \{1, ..., n\}$, $u_{ji} = u_{ij}$ (here and later we use the Einstein summation convention, so $u_i\,dx^i = \sum_{i=1}^n u_i\,dx^i$, etc.) A local diffeomorphism $\Delta : J^2(\pi) \to J^2(\pi)$, $\Delta : (x^i, u, u_i, u_{ij}) \mapsto (\bar{x}^i, \bar{u}, \bar{u}_i, \bar{u}_{ij})$, is called a *contact transformation* if for every contact 1-form $\bar{\vartheta}$ the form $\Delta^* \bar{\vartheta}$ is also contact. We denote by $\mathrm{Cont}(\mathrm{J}^2(\pi))$ the pseudo-group of contact transformations on $J^2(\pi)$.

Let $\mathcal{H}$ be a open subset of $\mathbb{R}^{(2n+1)(n+3)(n+1)/3}$ with local coordinates (a, b^i_k, c^i, f^{ik}, g_i, s_{ij}, w^k_{ij}, z_{ijk}), $i, j, k \in \{1, ..., n\}$, $i \le j$, such that $a \ne 0$, $\det(b^i_k) \ne 0$, $f^{ik} = f^{ki}$ and $z_{ijk} = z_{ikj} = z_{jik}$. Let (B^i_k) be the inverse matrix for the matrix (b^k_l), so $B^i_k\,b^k_l = \delta^i_l$. We consider the *lifted coframe*

$$\Theta_0 = a\,\vartheta_0, \quad \Theta_i = g_i\,\Theta_0 + a\,B^k_i\,\vartheta_k, \quad \Xi^i = c^i\,\Theta_0 + f^{ik}\,\Theta_k + b^i_k\,dx^k,$$
$$\Sigma_{ij} = s_{ij}\,\Theta_0 + w^k_{ij}\,\Theta_k + z_{ijk}\,\Xi^k + a\,B^i_k\,B^j_l\,du_{kl}, \tag{5}$$

defined on $J^2(\pi) \times \mathcal{H}$. As it is shown in [24], the forms (5) are MC forms for $\mathrm{Cont}(\mathrm{J}^2(\pi))$, that is, a local diffeomorphism $\widehat{\Delta} : J^2(\pi) \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$ satisfies the conditions $\widehat{\Delta}^*\,\bar{\Theta}_0 = \Theta_0$, $\widehat{\Delta}^*\,\bar{\Theta}_i = \Theta_i$, $\widehat{\Delta}^*\,\bar{\Xi}^i = \Xi^i$, and $\widehat{\Delta}^*\,\bar{\Sigma}_{ij} = \Sigma_{ij}$ if and only if it is projectable on $J^2(\pi)$, and its projection $\Delta : J^2(\pi) \to J^2(\pi)$ is a contact transformation.

The structure equations for $\mathrm{Cont}(\mathrm{J}^2(\pi))$ have the form

$$d\Theta_0 = \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,$$
$$d\Theta_i = \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik},$$
$$d\Xi^i = \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k,$$
$$d\Sigma_{ij} = \Phi^k_i \wedge \Sigma_{kj} - \Phi^0_0 \wedge \Sigma_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k,$$

where the additional forms Φ^0_0, Φ^0_i, Φ^k_i, Ψ^{i0}, Ψ^{ij}, Υ^0_{ij}, Υ^k_{ij}, and Λ_{ijk} depend on differentials of the coordinates of $\mathcal{H}$.

Suppose $\mathcal{E}$ is a second-order differential equation in one dependent and n independent variables. We consider $\mathcal{E}$ as a submanifold in $J^2(\pi)$. Let $\mathrm{Cont}(\mathcal{E})$ be the group of contact symmetries for $\mathcal{E}$. It consists of all the contact transformations on $J^2(\pi)$ mapping $\mathcal{E}$ to itself. Let $\iota_0 : \mathcal{E} \to J^2(\pi)$ be an embedding, and $\iota = \iota_0 \times \mathrm{id} : \mathcal{E} \times \mathcal{H} \to \mathrm{J}^2(\pi) \times \mathcal{H}$. The invariant 1-forms of $\mathrm{Cont}(\mathcal{E})$ are restrictions of the forms (5) to $\mathcal{E} \times \mathcal{H}$: $\theta_0 = \iota^*\Theta_0$, $\theta_i = \iota^*\Theta_i$, $\xi^i = \iota^*\Xi^i$, and $\sigma_{ij} = \iota^*\Sigma_{ij}$. The forms θ_0, θ_i, ξ^i, and σ_{ij} have some linear dependencies, i.e., there exists a non-trivial set of functions E^0, E^i, F_i, and G^{ij} on $\mathcal{E} \times \mathcal{H}$ such that $E^0\,\theta_0 + E^i\,\theta_i + F_i\,\xi^i + G^{ij}\,\sigma_{ij} \equiv 0$. These functions are lifted invariants of $\mathrm{Cont}(\mathcal{E})$. Setting them equal to some constants allows us to specify some coordinates a, b^k_i, c_i, g_i, f^{ij}, s_{ij}, w^k_{ij}, and z_{ijk} as functions of the coordinates on $\mathcal{E}$ and the other coordinates on $\mathcal{H}$.

After these normalizations, a part of the forms $\phi^0_0 = \iota^*\Phi^0_0$, $\phi^k_i = \iota^*\Phi^k_i$, $\phi^0_i = \iota^*\Phi^0_i$, $\psi^{ij} = \iota^*\Psi^{ij}$, $\psi^{i0} = \iota^*\Psi^{i0}$, $\upsilon^0_{ij} = \iota^*\Upsilon^0_{ij}$, $\upsilon^k_{ij} = \iota^*\Upsilon^k_{ij}$, and $\lambda_{ijk} = \iota^*\Lambda_{ijk}$, or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the coordinates on $\mathcal{H}$. Setting coefficients of the semi-basic forms equal to some constants, we get specifications of some more coordinates on $\mathcal{H}$.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

$$\begin{aligned}
d\theta_0 &= \phi^0_0 \wedge \theta_0 + \xi^i \wedge \theta_i,\\
d\theta_i &= \phi^0_i \wedge \theta_0 + \phi^k_i \wedge \theta_k + \xi^k \wedge \sigma_{ik},\\
d\xi^i &= \phi^0_0 \wedge \xi^i - \phi^i_k \wedge \xi^k + \psi^{i0} \wedge \theta_0 + \psi^{ik} \wedge \theta_k,\\
d\sigma_{ij} &= \phi^k_i \wedge \sigma_{kj} - \phi^0_0 \wedge \sigma_{ij} + \upsilon^0_{ij} \wedge \theta_0 + \upsilon^k_{ij} \wedge \theta_k + \lambda_{ijk} \wedge \xi^k.
\end{aligned}$$

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of MC forms for $\mathrm{Cont}(\mathcal{E})$. In the second case, when the reduced lifted coframe does not satisfy Cartan's test, we should use the procedure of prolongation, [27, ch 12].

3. Coverings of DEs

Let $\pi_\infty : J^\infty(\pi) \to \mathbb{R}^n$ be the infinite jet bundle of local sections of the bundle π. The coordinates on $J^\infty(\pi)$ are (x^i, u, u_I), where $I = (i_1, ..., i_k)$ are symmetric multi-indices, $i_1, ..., i_k \in \{1, ..., n\}$, and for any local section f of π there exists a section $j_\infty(f) : \mathbb{R}^n \to J^\infty(\pi)$ such that $u_I(j_\infty(f)) = \partial^{\#I}(f)/\partial x^{i_1} ... \partial x^{i_k}$, $\#I = \#(i_1, ..., i_k) = k$. Contact forms on $J^\infty(\pi)$ are defined by the requirement to satisfy $j_\infty(f)^*\,\vartheta = 0$ for any f. They are linear combinations of the forms $\vartheta_I = du_I - u_{Ii}\,dx^i$, $\#I \geq 0$. The *total derivatives* on $J^\infty(\pi)$ are defined in the local coordintes as

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\#I \geq 0} u_{Ii}\,\frac{\partial}{\partial u_I}.$$

We have $[D_i, D_j] = 0$ for $i, j \in \{1, ..., n\}$ and $\vartheta_I = D_I(\vartheta_0)$, where $D_I = D_{i_1} \circ ... \circ D_{i_k}$ for $I = (i_1, ..., i_k)$.

A differential equation $F(x^i, u, u_I) = 0$, $\#I \le q$, defines a submanifold

$$\mathcal{E}^\infty = \{D_K(F) = 0 \mid \#K \ge 0\} \subset J^\infty(\pi).$$

We denote restrictions of D_i and ϑ_I on $\mathcal{E}^\infty$ as $\bar{D}_i$ and $\bar{\vartheta}_I$, respectively.

In local coordinates, a *covering* over $\mathcal{E}^\infty$ is a bundle $\widetilde{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \mathcal{V} \to \mathcal{E}^\infty$ with fibre coordinates v^κ, $\kappa \in \{1, ..., N\}$ or $\kappa \in \mathbb{N}$, equipped with extended total derivatives

$$\widetilde{D}_i = \bar{D}_i + \sum_\kappa T_i^\kappa(x^j, u, u_I, v^\tau)\,\frac{\partial}{\partial v^\kappa}, \qquad i \in \{1, ..., n\},$$

such that $[\widetilde{D}_i, \widetilde{D}_j] = 0$ whenever $(x^i, u, u_I) \in \mathcal{E}^\infty$.

In terms of differential forms, the covering is defined by the forms

$$\widetilde{\vartheta}^\kappa = dv^\kappa - T_i^\kappa(x^j, u, u_I, v^\tau)\,dx^i$$

such that $d\widetilde{\vartheta}^\kappa \equiv 0 \pmod{\widetilde{\vartheta}^\tau, \bar{\vartheta}_I}$ whenever $(x^i, u, u_I) \in \mathcal{E}^\infty$. We call $\widetilde{\vartheta}^\kappa$ *Wahlquist–Estabrook forms* (WE forms) of the covering.

EXAMPLE. System (2), (3) provides an infinite-dimensional covering for (4) with fibre coordinates $v_0 = v$, $v_k = \partial^k v / \partial x^k$, $k \in \mathbb{N}$, the extended total derivatives

$$\widetilde{D}_t = \bar{D}_t + \sum_{j=0}^\infty \widetilde{D}_x^j((v_0^2 - u)\,v_1 - u_y - v_0\,u_x)\,\frac{\partial}{\partial v_j},$$

$$\widetilde{D}_x = \bar{D}_x + \sum_{j=0}^\infty v_{j+1}\,\frac{\partial}{\partial v_j},$$

$$\widetilde{D}_y = \bar{D}_y + \sum_{j=0}^\infty \widetilde{D}_x^j(v_0\,v_1 - u_x)\,\frac{\partial}{\partial v_j},$$

and the WE forms

$$\widetilde{\vartheta}_0 = dv_0 - ((v_0^2 - u)\,v_1 - u_y - v_0\,u_x)\,dt - v_1\,dx - (v_0\,v_1 - u_x)\,dy,$$
$$\widetilde{\vartheta}_k = \widetilde{D}_x^k(\widetilde{\vartheta}_0), \qquad k \in \mathbb{N}.$$

4. Symmetry pseudo-group and a covering for the modified Khokhlov - Zabolotskaya equation

By the method described in section 2 we compute MC forms and structure equations for the pseudo-group of contact symmetries of equation (1). The structure equations read

$$d\theta_0 = \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^1 \wedge \theta_2 + \xi_3 \wedge \theta_3,$$
$$d\theta_1 = \left(\tfrac{1}{2}\,\theta_2 + \xi^2\right) \wedge \theta_0 + \left(\tfrac{3}{2}\,\eta_1 + \xi^3 - \tfrac{3}{2}\,\sigma_{22}\right) \wedge \theta_1 + \left(\eta_1 + \theta_3 - \sigma_{22} + \xi^3\right) \wedge \theta_2 + 2\,\theta_3 \wedge \xi^2$$
$$\qquad + \xi^1 \wedge \sigma_{11} + (\xi^1 + \xi^2) \wedge \sigma_{12} + \xi^3 \wedge \sigma_{13},$$
$$d\theta_2 = \tfrac{1}{2}\,(\eta_1 - \sigma_{22}) \wedge \theta_2 + \xi^1 \wedge \sigma_{12} + (\xi^1 + \xi^2) \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23},$$

$$d\theta_3 = \tfrac{1}{2}\,\sigma_{22} \wedge \theta_0 - \xi^2 \wedge \theta_2 + \left(\eta_1 + \tfrac{1}{2}\,\xi^3 - \sigma_{22}\right) \wedge \theta_3 + \xi^1 \wedge \sigma_{13} + \xi^3 \wedge (\sigma_{12} + \sigma_{22})$$
$$+(\xi^1 + \xi^2) \wedge \sigma_{23},$$
$$d\xi^1 = -\tfrac{1}{2}\,\left(\eta_1 + 2\,\xi^3 - 3\,\sigma_{22}\right) \wedge \xi^1,$$
$$d\xi^2 = \left(\theta_3 - \tfrac{1}{2}\,\theta_0\right) \wedge \xi^1 + \tfrac{1}{2}\,(\eta_1 + \sigma_{22}) \wedge \xi^2 + (\theta_2 + \xi^2) \wedge \xi^3,$$
$$d\xi^3 = 2\,(\theta_2 + \xi^2) \wedge \xi^1 + \sigma_{22} \wedge \xi^3,$$
$$d\sigma_{11} = 2\,\eta_1 \wedge (\sigma_{11} + \sigma_{12}) + \eta_2 \wedge \xi^1 + \eta_3 \wedge (\xi^1 + \xi^2) + \eta_4 \wedge \xi^3 + \sigma_{22} \wedge \theta_0$$
$$+3\,(2\,\theta_2 - \sigma_{23}) \wedge \theta_1 + (3\,\sigma_{13} - 2\,\sigma_{23}) \wedge \theta_2 + (\theta_3 + 3\,\sigma_{11} + 2\,\sigma_{12}) \wedge \sigma_{22},$$
$$d\sigma_{12} = \eta_1 \wedge (\sigma_{12} + \sigma_{22}) + \eta_3 \wedge \xi^1 + \eta_5 \wedge \xi^3 + \tfrac{1}{2}\,\theta_0 \wedge \sigma_{22} + \tfrac{13}{2}\,\theta_1 \wedge \xi^1$$
$$+\tfrac{1}{2}\,\theta_2 \wedge (11\,\xi^1 + 3\,\xi^2 + 2\,\sigma_{23}) - \theta_3 \wedge \sigma_{22} - 2\,(2\,\sigma_{13} + \sigma_{23}) \wedge \xi^1 + 2\,\sigma_{12} \wedge \sigma_{22},$$
$$d\sigma_{13} = \tfrac{1}{2}\,\eta_1 \wedge (3\,\sigma_{13} + \sigma_{23}) + \eta_3 \wedge \xi^3 + \eta_4 \wedge \xi^1 + \eta_5 \wedge (\xi^1 + \xi^2) + \tfrac{1}{2}\,\theta_0 \wedge (3\,\theta_2 + 2\,\xi^2 - \sigma_{23})$$
$$+\tfrac{1}{2}\,\theta_1 \wedge (13\,\xi^3 - \sigma_{22}) + \tfrac{1}{2}\,\theta_2 \wedge (6\,\theta_3 + 13\,\xi^3 - 4\,\sigma_{12} - 2\,\sigma_{22}) - \theta_3 \wedge (4\,\xi^2 - \sigma_{23})$$
$$+(2\,\sigma_{11} + 3\,\sigma_{12} + \sigma_{22}) \wedge \xi^1 + (4\,\sigma_{12} + 3\,\sigma_{22}) \wedge \xi^2 + \tfrac{1}{2}\,xi^3\, \wedge (11\,\sigma_{13} + 6\,\sigma_{23})$$
$$-\tfrac{1}{2}\,\sigma_{22} \wedge (5\,\sigma_{13} + 2\,\sigma_{23}),$$
$$d\sigma_{22} = 2\,(2\,\theta_2 + 2\,\xi^2 - \sigma_{23}) \wedge \xi^1 + \tfrac{1}{2}\sigma_{22} \wedge \xi^3,$$
$$d\sigma_{23} = \tfrac{1}{2}\,\eta_1 \wedge \sigma_{23} + \eta_5 \wedge \xi^1 + \tfrac{3}{2}\,\theta_2 \wedge (2\,\xi^3 - \sigma_{22}) + \tfrac{1}{2}\,\xi^1 \wedge (3\,\sigma_{22} - 2\,\xi^3 - 2\,\sigma_{12})$$
$$+\tfrac{3}{2}\,\xi^2 \wedge (2\,\xi^3 - \sigma_{22}) + \tfrac{1}{2}\,(5\,\xi^3 - 3\,\sigma_{22}) \wedge \sigma_{23},$$
$$d\eta_1 = \xi^1 \wedge (\theta_2 + \xi^2) \wedge \xi^1 + \tfrac{1}{2}\,\xi^3 \wedge \sigma_{22},$$
$$d\eta_2 = \pi_1 \wedge \xi^1 + \pi_2 \wedge (\xi^1 + \xi^2) + \pi_3 \wedge \xi^3 + \tfrac{1}{2}\,\eta_1 \wedge (5\,\eta_2 + 6\,\eta_3 - 13\,\theta_1 + 16\,\theta_2 - 16\,\sigma_{13}$$
$$-8\,\sigma_{23}) + \tfrac{9}{2}\,\eta_2 \wedge \sigma_{22}\tfrac{1}{2}\,\eta_3 \wedge (2\,\theta_3 - \theta_0 + 6\,\sigma_{22}) + 5\,\eta_4 \wedge \theta_2 - \eta_5 \wedge (3\,\theta_1 + 2\,\theta_2)$$
$$-\theta_0 \wedge (16\,\theta_2 - 5\,\sigma_{23}) + \theta_1 \wedge (9\,\sigma_{12} + 14\,\sigma_{22}) + (32\,\theta_3 + 26\,\sigma_{11} + 5\,\sigma_{12} - 12\,\sigma_{22}) \wedge \theta_2$$
$$+10\,\sigma_{23} \wedge \theta_3 - 3\,(\sigma_{13} - 2\,\sigma_{23}) \wedge \sigma_{12} + \sigma_{23} \wedge (9\,\sigma_{11} - 4\,\sigma_{22}) + 10\,\sigma_{22} \wedge \sigma_{13},$$
$$d\eta_3 = \pi_2 \wedge \xi^1 + \pi_4 \wedge \xi^3 + \tfrac{1}{2}\,\eta_1 \wedge (6\,\theta_2 + 8\,\xi^1 + 8\,\xi^2 + 3\,\eta_3) + \tfrac{7}{2}\,\eta_3 \wedge \sigma_{22} - 2\,\eta_4 \wedge \xi^1$$
$$+3\,\eta_5 \wedge (\theta_2 + 2\,\xi^1 + 2\,\xi^2) + \tfrac{1}{2}\,\theta_0 \wedge (24\,\theta_2 + 41\,\xi^1 + 33\,\xi^2 - 3\,\sigma_{23}) + \tfrac{21}{2}\,\theta_1 \wedge \sigma_{22}$$
$$+6\,\theta_2 \wedge (4\,\theta_3 - 3\,\sigma_{12}) - \theta_3 \wedge (41\,\xi^1 - 33\,\xi^2 - 6\,\sigma_{23}) + 14\,\xi^1 \wedge \sigma_{11} + 20\,\sigma_{22} \wedge (\xi^1 + \xi^2)$$
$$+10\,\sigma_{12} \wedge (2\,\xi^1 + 3\,\xi^2) - 3\,\sigma_{12} \wedge \sigma_{23} - 3\,\sigma_{13} \wedge \sigma_{22},$$
$$d\eta_4 = \pi_2 \wedge \xi^3 + \pi_3 \wedge \xi^1 + \pi_4 \wedge (\xi^1 + \xi^2) + 2\,\eta_1 \wedge (\eta_4 + \eta_5 + 2\,\xi^3) + \eta_2 \wedge \xi^1$$
$$+\eta_3 \wedge (4\,\theta_2 + \xi^1 + \xi^2) + \tfrac{1}{2}\,\eta_4 \wedge (8\,\sigma_{22} - 21\,\xi^3) + 2\,\eta_5 \wedge (\sigma_{22} - \xi_3) + \tfrac{1}{2}\,\theta_0 \wedge (43\,\xi^3 - \sigma_{22})$$
$$+\tfrac{1}{2}\,\theta_1 \wedge (69\,\theta_2 - 3\,\xi^1 + 9\,\xi^2 - 9\,\sigma_{23}) + \tfrac{1}{2}\,\theta_2 \wedge (4\,\xi^1 + 12\,\xi^2 + 21\,\sigma_{13} + \sigma_{23})$$
$$-\theta_3 \wedge (43\,\xi^3 - \sigma_{22}) + \tfrac{1}{2}\,\xi^3 \wedge (67\,\sigma_{11} + 18\,\sigma_{12} - 28\,\sigma_{22}) + \tfrac{5}{2}\,\sigma_{11} \wedge \sigma_{22} - 6\,\sigma_{13} \wedge \sigma_{23},$$
$$d\eta_5 = \pi_4 \wedge \xi^1 + \tfrac{1}{2}\,\eta_1 \wedge (2\,\eta_5 + 2\,\xi^3 + \sigma_{22}) - 3\,\eta_3 \wedge \xi^1 + \tfrac{3}{2}\,\eta_5 \wedge (2\,\sigma_{22} - 2\,\xi^3)$$
$$-\tfrac{1}{4}\,\theta_0 \wedge (6\,\xi^3 - \sigma_{22}) - 26\,\theta_1 \wedge \xi^1 + \tfrac{1}{2}\,\theta_2 \wedge (5\,\sigma_{23} - 64\,\xi^1 - 3\,\xi^2) + \tfrac{1}{2}\,\theta_3 \wedge (6\,\xi^3 - \sigma_{22})$$
$$+\tfrac{1}{2}\,\xi^1 \wedge (12\,\xi^2 - 23\,\sigma_{13} - 19\,\sigma_{23}) + \tfrac{3}{2}\,\xi^2 \wedge \sigma_{23} + \tfrac{13}{2}\,\xi^3 \wedge (\sigma_{12} + 2\,\sigma_{22}) + \tfrac{5}{2}\,\sigma_{12} \wedge \sigma_{22}.$$

The forms η_1, ... , η_5 appear in the step of absorption of torsion in the reduced structure equations. We have

$$\xi^1 = q\,dt,$$

$$\xi_2 = u_{xx}^2\, q^{-1}\, \left(dx + u_x\, dy + \left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, dt\right),$$
$$\xi_3 = u_{xx}\, (2\, u_x\, dt + dy),$$
$$\eta_1 = 3\, (u_{xx})^{-1}\, du_{xx} - 2\, q^{-1}\, dq - \tfrac{1}{2}\, u_{xx}\, dy - u_x\, u_{xx}\, dt,$$

with $q = b_1^1 \neq 0$. We need not explicit expressions for the other MC forms in the sequel. We take a linear combination

$$\eta_1 + \xi^2 + \tfrac{1}{2}\, \xi^3 = 3\, (u_{xx})^{-1}\, du_{xx} - 2\, q^{-1}\, dq + \tfrac{1}{2}\, u_{xx}^2\, \left(\left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, dt + dx + u_x\, dy\right)$$

and substitute $u_{xx} = v^2\, v_1^2$, $q = \frac{1}{4}\, v^5\, v_1^3$. Then we have

$$\eta_1 + \xi^2 + \tfrac{1}{2}\, \xi^3 = -4\, v^{-1}\, \left(dv - \left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, v_1\, dt - v_1\, dx - u_x\, v_1\, dy\right).$$

This form annules whenever v satisfies the following system of PDEs:

$$v_t = \left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, v_1, \qquad v_x = v_1, \qquad v_y = u_x\, v_1. \tag{6}$$

Excluding v_1 from this system, we have the covering equations

$$v_t = \left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, v_x, \qquad v_y = u_x\, v_x. \tag{7}$$

Introducing fibre coordinates $v_0 = v$, $v_k = \partial^k\, v / \partial\, x^k$, $k \in \mathbb{N}$, we obtain from (6) the extended total derivatives

$$\widetilde{D}_t = \bar{D}_t + \sum_{j=0}^{\infty} \widetilde{D}_x^j(\left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, v_1)\, \frac{\partial}{\partial v_j},$$
$$\widetilde{D}_x = \bar{D}_x + \sum_{j=0}^{\infty} v_{j+1}\, \frac{\partial}{\partial v_j},$$
$$\widetilde{D}_y = \bar{D}_y + \sum_{j=0}^{\infty} \widetilde{D}_x^j(u_x\, v_1)\, \frac{\partial}{\partial v_j},$$

and the WE forms

$$\widetilde{\vartheta}_0 = dv_0 - \left(\tfrac{1}{2}\, u_x^2 + u_y\right)\, v_1\, dt - v_1\, dx - u_x\, v_1\, dy,$$
$$\widetilde{\vartheta}_k = \widetilde{D}_x^k(\widetilde{\vartheta}_0), \qquad k \in \mathbb{N}.$$

Excluding u from (7), we obtain

$$v_{yy} = v_{tx} + \frac{v_y^2 - v_t\, v_x}{v_x^2}\, v_{xx}. \tag{8}$$

Therefore, (7) is a Bäcklund transformation between equations (1) and (8).

5. Conclusion

We have shown that a covering for a nonlinear PDE in three independent variables can be revealed by means of Cartan's equivalence method. For the modified Khokhlov-Zabolotskaya equation (1) the covering equations appear from a linear combination of the MC forms of its symmetry pseudo-group. While there is the algorithm for computing the MC forms, further study is required to enlight the relation between Cartan's structure theory of symmetry pseudo-groups and nonlocal aspects of geometry of DEs.

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